MATH 2050C Mathematical Analysis I
2019-20 Term 2
Problem Set 3

due on Feb 21, 2020 (Friday) at 11:59PM

Instructions: You are allowed to discuss with your classmates or seek help from the TAs but you are required to write/type up your own solutions. Please submit your completed homework on time either during the tutorials or at the designated box (marked with the course number) outside the general office of Mathematics Department (Room 220 of Lady Shaw Building). **No late homework will be accepted.** Remember to write down your name (in English and Chinese) and student ID on your problem set. All the exercises below are taken from the textbook.

Required Readings: Chapter 2.4, 2.5

Optional Readings: Chapter 2.5 on binary and decimal representations, (Rudin Appendix of Chapter 1) Dedekind construction of $\mathbb{R}$

Problems to hand in

Section 2.4: Exercise # 4(b), 7, 11, 17

Section 2.5: Exercise # 8

Suggested Exercises

Section 2.4: Exercise # 1, 2, 3, 4(a), 5, 6, 8, 9, 10, 12, 14, 15, 16, 19

Section 2.5: Exercise # 2, 3, 7, 9, 10, 11

Challenging Exercises (optional)

1. (*Non-ordering of* $\mathbb{C}$) Prove that there is no ordering on $\mathbb{C}$ which makes it an ordered field.

2. (*Real exponential power*) Fix $b > 1$, show that one can define b^x for any $x \in \mathbb{R}$ as follow:

   (a) Let $r \in \mathbb{Q}$. Suppose $r = \frac{m}{n} = \frac{p}{q}$ such that $m, n, p, q \in \mathbb{Z}$, and $n, q > 0$. Prove that $(b^m)^{1/n} = (b^p)^{1/q}$. Hence b^r is well-defined for $r \in \mathbb{Q}$.

   (b) Prove that $b^r b^s = b^{r+s}$ for all $r, s \in \mathbb{Q}$.

   (c) Fix $x \in \mathbb{R}$. Define $B := \{b^t : t \in \mathbb{Q} \text{ and } t \leq x\}$. Prove that $b^x = \sup B$ when $x \in \mathbb{Q}$. Therefore, it makes sense to define $b^x = \sup B$ for any $x \in \mathbb{R}$.

   (d) Prove that $b^x b^y = b^{x+y}$ for any $x, y \in \mathbb{R}$.

3. (*Logarithm with base* b) Fix $b > 0$, $y > 0$, show that there is a unique $x \in \mathbb{R}$ such that $b^x = y$ by following the steps below:

   (a) For any $n \in \mathbb{N}$, show that $b^n - 1 \geq n(b-1)$. Hence, $b - 1 \geq n(b^{1/n} - 1)$.

   (b) If $t > 1$ and $n > \frac{b-1}{t-1}$, prove that $b^{1/n} < t$.

   (c) Suppose $w \in \mathbb{R}$ satisfies $b^w < y$. Show that $b^{w+\frac{1}{n}} < y$ for any sufficiently large $n \in \mathbb{N}$.

   (d) Suppose $w \in \mathbb{R}$ satisfies $b^w > y$. Show that $b^{w-\frac{1}{n}} > y$ for any sufficiently large $n \in \mathbb{N}$.

   (e) Define $A := \{w \in \mathbb{R} \ : \ b^w < y\}$. Show that $x := \sup A$ satisfies $b^x = y$. Prove that such an $x \in \mathbb{R}$ is unique.